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Best Rational Product Approximations of Functions

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1. INTRODUCTION

Let $f(x, y) \in C[D]$, where $D = [a, b] \times [c, d]$, and let $G(C, x, y)$ be continuous on $\mathscr{P} \times D$, where $\mathscr{P} \subseteq E_n$. Then the classical approximation problem on the region D is that of insuring the existence of a $C^* \in \mathscr{P}$ such that $G(C^*, x, y)$ satisfies

$$\sup_D | G(C^*, x, y) - f(x, y)| = \inf_{\mathscr{P}} \sup_D | G(C, x, y) - f(x, y)|.$$

Even though such a C^* may exist, $G(C^*, x, y)$ may not be unique (see [4]).

Weinstein [5, 6] defines a *unique best product approximation to* $f(x, y)$ *with respect to* y in the following manner. Let $\{\phi_i(x)\}_{i=1}^n$ and $\{\psi_j(y)\}_{j=1}^m$ be Chebyshev sets on $I = [a, b]$ and $J = [c, d]$, respectively. For each $\alpha = (a_1, a_2, ..., a_n) \in E_n$, let $P_\alpha = a_1\phi_1 + a_2\phi_2 + \cdots + a_n\phi_n$. Define $f_y(x) = f(x, y)$ for fixed y. Then $f_y(x)$ is continuous on I, and, hence, for each y there exists a unique polynomial

$$P_{\alpha(y)} = \sum_{i=1}^n a_i(y)\, \phi_i(x)$$

that satisfies

$$\inf_{\alpha \in E_n} \sup_I \left| f_y(x) - \sum_{i=1}^n a_i\phi_i(x) \right| = \sup_I \left| f_y(x) - \sum_{i=1}^n a_i(y)\, \phi_i(x) \right|.$$

If the functions $a_i(y)$, $i = 1,..., n$, are continuous on J, then for each i there exists a unique polynomial $Q_{\alpha_i} = \sum_{j=1}^m a_{ij}\psi_j(y)$ that best approximates $a_i(y)$ on J in the Chebyshev sense. Then

$$T_A(x, y) = \sum_{i=1}^n \sum_{j=1}^m a_{ij}\psi_j(y)\, \phi_i(x)$$

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is defined to be the best product Chebyshev approximation to $f(x, y)$ relative to the variable y. When the approximating sets $\{\phi_i(x)\}_{i=1}^n$ and $\{\psi_j(y)\}_{j=1}^m$ are Chebyshev sets, it can be shown that the functions $a_i(y)$, $i = 1, 2,..., n$, are continuous (see [5]) and that $T_A(x, y)$ is uniquely determined for each $f \in C[D]$.

2. Rational Functions

In this note we discuss possible extensions of the idea of best product approximation. Let

$$R(C, x) = \frac{N(A, x)}{D(B, x)} = \frac{\sum_{i=0}^n a_i x^i}{\sum_{i=0}^m b_i x^i},$$

where $C = (A; B) = (a_0, a_1,..., a_n; b_0, b_1,..., b_m)$ satisfies

(i) $D(B, x) > 0$ for all $x \in I$,

(ii) $N(A, x)$ and $D(B, x)$ have no common factors other than constants and $|a_i| > 0$ for some i, $0 \leqslant i \leqslant n$, and

(iii) $\sum_{j=0}^m b_j^2 = 1$.

Let $\mathscr{P}$ consist of all vectors in E_{m+n+2} that satisfy conditions (i), (ii), and (iii), and the vector $C_0 = (0, 0,..., 0; 1, 0,..., 0)$. Then $R(C_0, x) \equiv 0$. Thus, the zero rational function is uniquely represented in $\mathscr{P}$ by the vector C_0. We note that if $R(C_1, x) \equiv R(C_2, x)$ on I and if C_1 and C_2 are in $\mathscr{P}$, then $C_1 = C_2$. It is well known (see [1, 3]) that if $f \in C[I]$, then there exists a unique element $C^* \in \mathscr{P}$ such that

$$\sup_I |f(x) - R(C^*, x)| = \inf_{\mathscr{P}} \sup_I |f(x) - R(C, x)|.$$

Initially we are interested in the following problem. Let $f \in C[D]$ and define $f_y(x) = f(x, y)$. Let $R(C(y), x)$ be the best approximation to $f_y(x)$ in the sense that

$$\inf_{C \in \mathscr{P}} \sup_I |f_y(x) - R(C, x)| = \sup_I |f_y(x) - R(C(y), x)|.$$

If $C(y) = (a_0(y), a_1(y),..., a_n(y); b_0(y), b_1(y),..., b_m(y))$ is continuous in y, where

$$\| C(y)\|^2 = \sum_{i=0}^n a_i^2(y) + \sum_{i=0}^m b_i^2(y),$$

(and consequently a_i and b_k are elements of $C[J]$, $0 \leqslant i \leqslant n$, $0 \leqslant k \leqslant m$) then we best approximate $a_i(y)$ and $b_k(y)$ in the Chebyshev sense on J by the approximating functions

$$N_{A_i}(y) = \sum_{j=0}^{l} a_{ij} y^j \qquad \text{and} \qquad N_{B_k}(y) = \sum_{j=0}^{l} b_{kj} y^j$$

or the approximating functions

$$N^*_{A_i}(y) = \frac{\sum_{j=0}^{l} a^*_{ij} y^j}{\sum_{j=0}^{s} a^{**}_{ij} y^j} \qquad \text{and} \qquad D^*_{B_k}(y) = \frac{\sum_{j=0}^{l} b^*_{kj} y^j}{\sum_{j=0}^{s} b^{**}_{kj} y^j}.$$

Then either

$$T_P(x, y) = \frac{\sum_{i=0}^{n} \sum_{j=0}^{l} a_{ij} x^i y^j}{\sum_{i=0}^{m} \sum_{j=0}^{l} b_{ij} x^i y^j} \tag{1}$$

or

$$T_R(x, y) = \frac{\sum_{i=0}^{n} N^*_{A_i}(y)\, x^i}{\sum_{i=0}^{m} D^*_{B_i}(y)\, x^i} \tag{2}$$

is the best rational product approximation (Chebyshev sense) with respect to y to $f(x, y)$ on D, provided that these expressions are meaningful. The functions $T_P(x, y)$ and $T_R(x, y)$ will be uniquely determined if $C(y)$ is continuous and the denominators do not vanish on D.

The following example demonstrates that even for very simple functions $f(x, y)$, $C(y)$ may not be continuous. Let $f(x, y) = x + y$, $D = [-1, 1] \times [0, 1]$. Then the best approximation to $f_y(x)$ on $[-1, 1]$ with coefficients in $\mathscr{P} = \{(a_0\,; b_0\,, b_1)\}$ is

$$R(C(y), x) = \frac{y^2/(y^2 + 2)^{1/2}}{(y^2 + 1)^{1/2}/(y^2 + 2)^{1/2} - x/(y^2 + 2)^{1/2}} \qquad \text{for} \quad y \neq 0.$$

If $y = 0$, $R(C(0), x) \equiv 0$. That is, for $y \neq 0$,

$$C(y) = \left(\frac{y^2}{(y^2 + 2)^{1/2}}\,; \frac{(y^2 + 1)^{1/2}}{(y^2 + 2)^{1/2}}, -\frac{1}{(y^2 + 2)^{1/2}}\right).$$

But the representation of $R(C(0), x) \equiv 0$ in $\mathscr{P}$ is $C(0) = (0; 1, 0)$. Hence, $\lim_{y \to 0} C(y) \neq C(0)$. In order to avoid the difficulties of this example, we shall restrict the class of functions from $C[D]$ that are to be approximated. We shall employ the following standard definitions (see [1–3]).

DEFINITION 1. A rational function $R(C, x)$ is said to be of degree $m(C) = n + m - d + 1$ at $C \in \mathscr{P}$ if $R(C, x)$ may be written as

$$R(C, x) = \frac{a_0 + a_1 x + \cdots + a_{n-p} x^{n-p}}{b_0 + b_1 x + \cdots + b_{m-q} x^{m-q}},$$

where $d = \min[p, q]$ and where $a_{n-p} \neq 0$ and $b_{m-q} \neq 0$. If $R(C, x) \equiv 0$, then $m(C) = n + 1$.

DEFINITION 2. The set of rational functions of degree at most n for the numerator and of degree at most m for the denominator with coefficients in $\mathscr{P}$ is denoted by $R(n, m)$.

DEFINITION 3. Let $R(C_f, x)$ be the best rational approximation to $f(x)$ from $R(n, m)$ on I. Then the function $f(x)$ is normal for (n, m) if $m(C_f) = n + m + 1$.

In the theorem below and the remainder of the paper, the following notation is employed; $\|\cdot\|_I = \sup_I |\cdot|$, $\|\cdot\|_J = \sup_J |\cdot|$, $\|\cdot\|_D = \sup_D |\cdot|$, and $\|\cdot\|$ is as previously defined.

THEOREM 1. *Suppose that for fixed $y^* \in J$, $f(x, y^*)$ is normal on I. Let $R(C(y), x)$ be the best rational approximation to $f_y(x)$ from $R(n, m)$ for each y. Then the function $C(y)$ is continuous at y^*.*

Proof. Let $\epsilon > 0$ be given. If y^* is not an endpoint of J, we need to show that there exists a $\delta > 0$ such that $|y - y^*| < \delta$ implies that $\| C(y) - C(y^*)\| < \epsilon$. If no such δ exists, then there exists a null sequence $\{\delta_n\}$ and a set $\{y_n\} \subseteq J$ such that $\| C(y_n) - C(y^*)\| \geqslant \epsilon$ and such that $|y^* - y_n| < \delta_n$. Let

$$\rho(y) = \inf_{C \in \mathscr{P}} \sup_I |f_y(x) - R(C, x)| = \inf_{C \in \mathscr{P}} \|f_y(x) - R(C, x)\|_I. \tag{3}$$

Then

$$\rho(y) = \|f_y(x) - R(C(y), x)\|_I, \tag{4}$$

and $\rho(y)$ is continuous on J. For all n,

$$\| R(C(y_n), x) - f_{y^*}(x)\|_I \leqslant \| R(C(y_n), x) - f_{y_n}(x)\|_I + \|f_{y_n}(x) - f_{y^*}(x)\|_I. \tag{5}$$

Inequality (5) implies that

$$\| R(C(y_n), x) - f_{y^*}(x)\|_I \leqslant \rho(y_n) + \sup_D |f(x, y_n) - f(x, y^*)|.$$

Thus,

$$\rho(y^*) \leqslant \| R(C(y_n), x) - f_{y^*}(x)\|_I \leqslant \rho(y_n) + \| f(x, y_n) - f(x, y^*)\|_D .$$

Therefore,

$$\lim_{n\to\infty} \| R(C(y_n), x) - f_{y^*}(x) \|_I = \rho(y^*). \tag{6}$$

For all $C \in \mathscr{P}$, $| R(C, x)| \leqslant M$ implies that $\| C \| \leqslant N$, where M and N are positive constants. Therefore, (6) implies that $\{\| C(y_n)\|\}$ is a uniformly bounded sequence. Hence, there exists a subsequence $\{C_{n_j}\}$ converging to $\bar{C} \in E_{m+n+2}$. (We note that $\mathscr{P}$ is not necessarily closed for $m \geqslant 1$.) Let $R(C^*, x)$ be the element in $R(n, m)$ associated with $\bar{C}$. That is, $\lim_{j\to\infty} R(C_{n_j}, x)$ exists at all but possibly a finite number of points, and there exists a rational function $R(C^*, x)$, $C^* \in \mathscr{P}$ that agrees with $\lim_{j\to+\infty} R(C_{n_j}, x)$ except at possibly a finite number of points (see [1, p. 77]). Then $R(C^*, x) \equiv R(C(y^*), x)$ by uniqueness of best approximations from $R(n, m)$, and, hence, $C^* = C(y^*)$. Thus, if $\bar{C} \in \mathscr{P}$, $\bar{C} = C(y^*)$. Suppose that $\bar{C} \in \bar{\mathscr{P}} - \mathscr{P}$. (We are assuming $m > 0$ since for $m = 0$, $\mathscr{P}$ is closed.) Let $\bar{C} = (\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_n; \bar{b}_0, \bar{b}_1, \ldots, \bar{b}_m)$. Then either $N(\bar{A}, x) = \bar{a}_0 + \bar{a}_1 x + \cdots + \bar{a}_n x^n$ and $D(\bar{B}, x) = \bar{b}_0 + \bar{b}_1 x + \cdots + \bar{b}_m x^m$ have common factors or $N(\bar{A}, x) \equiv 0$. This implies that $m(C(y^*)) < m + n + 1$, contradicting the normality of $f_{y^*}(x)$ on I. Therefore, $\bar{C} \in \mathscr{P}$ and $\bar{C} = C(y^*)$. Thus,

$$0 = \lim_{j\to\infty} \| C(y_{n_j}) - C(y^*)\| \geqslant \epsilon,$$

a contradiction. Hence, $C(y)$ is continuous at y^*.

It is apparent that if y^* is an endpoint of J, a slight modification of the above argument establishes continuity (from the right or left) of $C(y)$ at y^*.

COROLLARY. *Let $C(y) = (a_0(y), \ldots, a_n(y); b_0(y), \ldots, b_m(y))$ be as in Theorem 1. Then the functions $a_i(y)$, $i = 0, 1, \ldots, n$, and $b_j(y)$, $j = 0, 1, \ldots, m$ are continuous at y^*.*

3. THE BEST RATIONAL PRODUCT APPROXIMATION

We are now in a position to define more precisely the best rational product approximation with respect to y to the function $f(x, y)$ on D. The definition is given only in terms of polynomial approximations to $a_i(y)$ and $b_i(y)$; a similar definition is evident from the remarks preceding Eq. (2) in the case that rational functions in y are used to approximate $a_i(y)$ and $b_i(y)$.

Suppose that $R(C(y), x)$ is the best rational approximation from $R(n, m)$

to $f_y(x)$ on I, and that $f_y(x)$ is normal for (n, m) on I for each fixed $y \in J$. Let $N_i(A, y) = \sum_{j=0}^{l} a_{ij} y^j$ be the best polynomial approximation (Chebyshev sense) of degree l to $a_i(y)$ on J, and let $D_i(B, y) = \sum_{j=0}^{l} b_{ij} y^j$ be the best polynomial approximation to $b_i(y)$ on J. Since $b_0(y) + b_1(y)x + \cdots + b_m(y)\, x^m > 0$, we select l large enough to insure that $\sum_{i=0}^{m} D_i(B, y)\, x^i > 0$. Then the best rational product approximation with respect to y to $f(x, y)$ on D is

$$T_P(x, y) = \frac{\sum_{i=0}^{n} \sum_{j=0}^{l} a_{ij} y^j x^i}{\sum_{i=0}^{m} \sum_{j=0}^{l} b_{ij} y^j x^i}\,.$$

Remark. Suppose that $f(x)$ is normal for (n, m) on I, and that $g(y) \neq 0$ for all $y \in J$. Then if $R(C, x)$ is the best approximation from $R(n, m)$ to $f(x)$ on I, and if $P(A, y)$ is the best polynomial approximation of degree l to $g(y)$ on J, then $R(C(y), x) = R(C, x)\, P(A, y)$ is the best rational product approximation with respect to y to $f(x, y) = f(x)\, g(y)$ on D.

Example. Let $f(x, y) = xy + 3x + y + 3$, $D = [-1, 1] \times [-1, 1]$, and $\mathscr{P} = \{(a_0\,; b_0\,, b_1)\}$. For this $f(x, y)$, $a_0(y) = (y + 3)/\sqrt{3}$, $b_0(y) = \sqrt{2}/\sqrt{3}$, and $b_1(y) = -1/\sqrt{3}$. Then for $l \geqslant 1$, (1) is $T_P(x, y) = (y + 3)/(\sqrt{2} - x)$. If $a_0(y)$, $b_0(y)$, and $b_1(y)$ are best approximated by elements of $R(0, 1)$, then (2) becomes $T_R(x, y) = 9/(2\sqrt{5} - \sqrt{2}y - \sqrt{10}x + xy)$.

DEFINITION 4. If for all $N > 0$, there exists an $n > N$ and a corresponding $m(n)$ such that $f(x)$ is normal for $(n, m(n))$ on I, we say that $f(x)$ is normal on I for arbitrarily large n.

We note that if $f(x)$ is any continuous function on I, then f is normal for arbitrarily large n, since every continuous function is normal for $(n, 0)$. Let $f(x, y)$ be continuous on D. Then $f(x, y)$ can be arbitrarily closely approximated in the sense of the norm by an appropriate $T_P(x, y)$ or a $T_R(x, y)$. We outline the proof for these assertions. Let

$$e(a_i\,, J) = \left\| a_i(y) - \sum_{j=0}^{l} a_{ij} y^j \right\|_J .$$

$$e(b_i\,, J) = \left\| b_i(y) - \sum_{j=0}^{l} b_{ij} y^j \right\|_J ,$$

and

$$e_{l(n,m)}(A, B; J) = \max \left\{ \max_{0 \leqslant i \leqslant n} e(a_i\,, J), \quad \max_{0 \leqslant i \leqslant m} e(b_i\,, J) \right\}.$$

Then if $f(x, y)$ is normal for (n, m) on I for each fixed $y \in J$, it can be shown that

$$\|f(x, y) - T_P(x, y)\|_D \leqslant E_{n,m}(f, D) + \theta(m, n)\, e_{l(n,m)}(A, B; J), \tag{7}$$

where $E_{n,m}(f, D) = \sup_{y\in J} \rho(y)$ and where $\theta(m, n)$ is a nonnegative function in m and n. If $\epsilon > 0$ is given, then for n sufficiently large, $E_{n,m}(f, D) < \epsilon/2$, where n and m are selected such that f is normal for (n, m). The proof of this remark parallels that given in [5, p. 444]. For this fixed n and m, select l to insure that $T_P(x, y)$ exists and to insure that

$$\theta(m, n)\, e_{l(m,n)}(A, B; J) < \epsilon/2.$$

Thus, (7) implies that for the $T_P(x, y)$ corresponding to (n, m) and l

$$\|f(x, y) - T_P(x, y)\|_D < \epsilon.$$

A similar but slightly more complex result is easily obtained for $T_R(x, y)$.

4. Conclusions

It would appear that one could further extend the above results by best approximating $f_y(x)$ on $[a, b]$ by nonlinear approximating functions $G(A, x)$, $A \in \mathscr{P} \subseteq E_m$. The main problem is again to show that if

$$\|f_y(x) - G(A(y), x)\| = \inf_{A\in\mathscr{P}} \sup_{I} |f_y(x) - G(A, x)|,$$

then $A(y) = (a_1(y), a_2(y),..., a_m(y))$ is continuous on J. If this result is obtained, then one best approximates $a_i(y)$ by some suitable approximating function $H_i(B_i, y)$, $B_i \in Q \subseteq E_n$. The best product approximation relative to y on D is then $G[H(B, y), x]$, where $H(B, y) = (H_1(B_1, y), H_2(B_2, y),..., H_m(B_m, y))$; based on these remarks, perhaps this type of approximation would be more appropriately entitled "best composite approximation" to $f(x, y)$ relative to y on D.

References

1. J. R. Rice, "The Approximation of Functions," Vol. 1, Addison Wesley, Reading, MA, 1964.
2. J. R. Rice, "The Approximation of Functions," Vol. 2, Addison Wesley, Reading, MA, 1969.

3. T. J. RIVLIN, "An Introduction to the Approximation of Functions," Blaisdell Publishing Co., Waltham, MA, 1969.
4. T. J. RIVLIN AND H. S. SHAPIRO, Some uniqueness problems in approximation theory, *Comm. Pure Appl. Math.* **13** (1960), 35–47.
5. S. E. WEINSTEIN, Approximations of functions of severai variables: product Chebychev approximations, I, *J. Approximation Theory* **2** (1969), 433–447.
6. S. E. WEINSTEIN, Product approximations of functions of several variables, *SIAM J. Numer. Anal.* **8** (1971), 178–189.